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The superharmonic normal mode instabilities of nonlinear deep-water capillary waves

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We present results of the calculation of superharmonic normal mode perturbations to the exact nonlinear deep-water capillary wave solution of Crapper (1957). By using the method of Longuet-Higgins (1978*a*), we are able for the first time to consider all waveheights up to and including the maximum for two-dimensional perturbations. We find agreement with the recent asymptotic analysis of Hogan, Gruman & Stiassnie (1988). Superharmonic instabilities are found at various waveheights less than the maximum.

1. Introduction

In this paper we investigate the stability to superharmonic normal-mode perturbations of the exact nonlinear deep-water capillary wave solution of Crapper (1957). This celebrated closed solution satisfies exactly the nonlinear boundary conditions of wave motion on the surface of an ideal fluid under the influence of the restoring force of surface tension only. As is well known it differs from the solution for gravity waves in that it need not be expressed as an infinite Fourier series.

The calculation of the stability of this solution to three-dimensional perturbations has been attempted before by Chen & Saffman (1985), using the method of McLean *et al.* (1981). But the numerical computations proved unreliable at a value of the wave steepness far short of the maximum. In that calculation, the exact solution was inverted and expressed as a Fourier series. In this paper we restrict our attention to two-dimensional perturbations to the waveform but can consider arbitrary waveheights by using the method of Longuet-Higgins (1978*a*,*b*). In addition we use the exact form of Crapper's (1957) solution and so avoid any singularities associated with inversion, as found by Chen & Saffman (1985). Our results are shown to agree very well with the asymptotic analysis of Hogan, Gruman & Stiassnie (1988), namely that increasing the waveheight reduces the relative speed of the perturbations. At certain waveheights, two eigenvalues collide to produce the new type of superharmonic instability discovered by MacKay & Saffman (1986) for gravity waves. We shall consider subharmonic normal-mode perturbations in another paper.

In §2, we give an outline of the method of Longuet-Higgins (1978*a*), as applicable to any free-surface flow under the action of surface tension, in the absence of viscosity. In §3, we consider the basic flow to be a uniform flow and show that the perturbations are simply free capillary waves travelling with or against the stream.

In §4, we introduce Crapper's (1957) exact solution and in §5 we give the form of the normal-mode perturbations that we shall consider. We note that there we use Tanaka's rearrangement technique as given in Longuet-Higgins (1986). This is a simple operation on two square matrices which essentially halves the amount of computing time required. Section 6 contains the results together with comparison with the theory of Hogan *et al.* (1987). We summarize our conclusions in §7.

2. Nonlinear boundary conditions and perturbation analysis

We take an arbitrary time-dependent, two-dimensional and irrotational flow of a fluid which is incompressible and inviscid. We include the restoring force of surface tension and neglect gravity. For the moment, it is not necessary to assume a wave-like flow. The velocity potential and stream function are denoted by $\phi(x,y,t)$ and $\psi(x,y,t)$ respectively. x and y are Cartesian coordinates, with y vertically downwards, x to the left, and t is time. At the free surface $\mathscr{F}(x,y,t)=0$, we must satisfy the condition that particles initially at the surface remain there, that is

$$\frac{\mathrm{D}\mathscr{F}}{\mathrm{D}t}=0 \quad \text{on } \mathscr{F}=0 \tag{2.1}$$

together with Bernoulli's condition

$$\tfrac{1}{2}\boldsymbol{q}^2+\frac{\tau}{R}+\frac{\partial\phi}{\partial t}=\beta(t) \quad \text{on } \mathscr{F}=0, \tag{2.2}$$

where $\boldsymbol{q}=\boldsymbol{\nabla}\phi$, τ is the surface tension divided by the density, R is the radius of curvature of the surface, and β is independent of position.

Much of this section closely parallels §2 of Longuet-Higgins (1978*a*). We shall merely quote relevant results from that paper. In particular, since

$$\frac{\partial\phi}{\partial t}=q^2\frac{\partial(x,y)}{\partial(\psi,t)};\quad \frac{\partial\psi}{\partial t}=-q^2\frac{\partial(x,y)}{\partial(\phi,t)};\quad q^2=\frac{\partial(\phi,\psi)}{\partial(x,y)}$$

then (2.2) becomes

$$\frac{\partial(x,t)}{\partial(\psi,t)}+\left\{\frac{\tau}{R}-\beta\right\}\frac{\partial(x,y)}{\partial(\phi,\psi)}+\tfrac{1}{2}=0 \quad \text{on } \mathscr{F}=0. \tag{2.3}$$

If we take

$$\mathscr{F}(x,y,t)=G(\phi,\psi,t)\equiv\psi-\epsilon F(\phi,t), \tag{2.4}$$

where ϵ is a small parameter on the scales of the unperturbed flow, then the kinematic condition (2.1) can be written as

$$\epsilon\left(1+\frac{\partial(x,y)}{\partial(\psi,t)}\right)F_\phi+\epsilon\frac{\partial(x,y)}{\partial(\phi,\psi)}F_t+\frac{\partial(x,y)}{\partial(\phi,t)}=0 \quad \text{on } G=0. \tag{2.5}$$

This is, of course, identical with equation (2.7) of Longuet-Higgins (1978*a*) since surface tension plays no explicit part in (2.1).

In order to calculate the form of (2.3) in the light of (2.4), we first consider the radius-of-curvature term. Thus

$$\frac{1}{R}=\frac{2G_xG_yG_{xy}-G_{yy}G_x^2-G_{xx}G_y^2}{(G_x^2+G_y^2)^{\frac{3}{2}}}. \tag{2.6}$$

Then using the Cauchy–Riemann equations for (x,y), regarded as functions of (ϕ,ψ), we find that (2.3) becomes

$$-(y_\phi y_t+y_\psi x_t)+\tfrac{1}{2}+\left\{\tau\left[\frac{2G_xG_yG_{xy}-G_{yy}G_x^2-G_{xx}G_y^2}{(G_x^2+G_y^2)^{\frac{3}{2}}}\right]-\beta\right\}(y_\psi^2+y_\phi^2)=0, \tag{2.7}$$

and we can rewrite (2.5) as

$$\epsilon[1-(y_\phi y_t+y_\psi x_t)]F_\phi+\epsilon(y_\phi^2+y_\psi^2)F_t+(y_\psi y_t-y_\phi x_t)=0. \tag{2.8}$$

Equations (2.7) and (2.8) hold for any flow, subject to the constraints mentioned in the first paragraph of this section.

We now perturb (2.7) and (2.8) about a steady flow whose free surface is given by $\psi=0$. We take, in general, therefore for two-dimensional perturbations

$$\left.\begin{aligned} x(\phi,\psi,t) &= X(\phi,\psi)+\epsilon\xi(\phi,\psi,t),\\ y(\phi,\psi,t) &= Y(\phi,\psi)+\epsilon\eta(\phi,\psi,t),\end{aligned}\right\} \tag{2.9}$$

where ξ, η and F are all of order 1. We then substitute (2.9) into the governing equations (2.7) and (2.8), and Taylor expand each term about $\psi=0$.

The Bernoulli equation (2.7) gives us at $O(1)$

$$\tfrac{1}{2}+(Y_\phi^2+Y_\psi^2)\left(\frac{\tau}{R}-\beta\right)=0 \quad \text{on } \psi=0, \tag{2.10}$$

and at $O(\epsilon)$

$$\left.\begin{aligned} &-(Y_\phi\eta_t+Y_\psi\xi_t)+[2\eta_\phi Y_\phi+2\eta_\psi Y_\psi+(Y_\phi^2+Y_\psi^2)_\psi F]\left(\frac{\tau}{R}-\beta\right)\\ &\quad+\tau(Y_\phi^2+Y_\psi^2)\left\{qF_{\phi\phi}-q_\phi F_\phi+\left(\frac{1}{R}\right)_\psi F+q^5[(\eta_\psi Y_{\phi\phi}+\eta_{\phi\phi}Y_\psi\right.\\ &\quad\left.-\eta_\phi Y_{\phi\psi}-\eta_{\phi\psi}Y_\phi)(Y_\phi^2+Y_\psi^2)-3(\eta_\phi Y_\phi+\eta_\psi Y_\psi)(Y_\psi Y_{\phi\phi}-Y_\phi Y_{\phi\psi})]\right\}=0 \quad \text{on } \psi=0.\end{aligned}\right. \tag{2.11}$$

The kinematic condition is satisfied identically at $O(1)$. At $O(\epsilon)$ we have

$$F_\phi+(Y_\psi^2+Y_\phi^2)F_t+(Y_\psi\eta_t-Y_\phi\xi_t)=0 \quad \text{on } \psi=0. \tag{2.12}$$

We seek wave-like forms for ξ, η and F with time-dependence $e^{-i\sigma t}$ where $\operatorname{Im}(\sigma)>0$. The fastest growing instability will have the largest $\operatorname{Im}(\sigma)$. The foregoing analysis is valid for arbitrary well-behaved steady flows $x=X(\phi,\psi)$; $y=Y(\phi,\psi)$.

3. The perturbation of a uniform flow

The equations in §2 are quite general and we have at our disposal a simple flow with which to check our calculations. We consider the uniform flow given by

$$X=\phi, \quad Y=\psi. \tag{3.1}$$

Equation (2.10) is then satisfied provided $\beta=\frac{1}{2}$.

From (2.11) with this value of β, we find

$$-\xi_t-\eta_\psi+\tau(F_{\phi\phi}+\eta_{\phi\phi})=0, \tag{3.2}$$

and from (2.12) we have

$$F_\phi+F_t+\eta_t=0. \tag{3.3}$$

For solutions of the form

$$\left.\begin{aligned} \xi &= i\eta = a\,e^{-k(\psi-i\phi)-i\sigma t},\\ F &= -b\,e^{ik\phi-i\sigma t},\end{aligned}\right\} \tag{3.4}$$

we obtained the coupled equations

$$\left.\begin{aligned} (k-\sigma-\tau k^2)a+\tau k^2 b &= 0,\\ \sigma a+(k-\sigma)b &= 0.\end{aligned}\right\} \tag{3.5}$$

For non-zero values of a and b, we require

$$\sigma = k \pm (\tau k^3)^{\frac{1}{2}}, \tag{3.6}$$

which is the correct phase speed for free capillary waves travelling with or against the stream.

In the next section, we consider the unperturbed flow to be a uniform nonlinear wavetrain.

4. Pure capillary waves

The steady solution X, Y that we shall consider in the rest of the paper is originally due to Crapper (1957), who calculated the explicit exact solution for nonlinear pure capillary waves on water of infinite depth.

We view the wavetrain in the frame of reference moving with phase speed c. The wavelength is λ and the crest-to-trough waveheight is $2a$. Crapper found that

$$\left.\begin{aligned} X(\phi,\psi) &= \frac{\phi}{c} - \frac{4B\sin\theta}{k[1+B^2+2B\cos\theta]}, \\ Y(\phi,\psi) &= \frac{\psi}{c} + \frac{4B(B+\cos\theta)}{k[1+B^2+2B\cos\theta]}, \end{aligned}\right\} \tag{4.1}$$

where k is the wavenumber, $\theta = k\phi/c$, $B = A\,\mathrm{e}^{-k\psi/c}$, A is a monotonic function of the wave steepness ak given by

$$ak = \frac{4A}{(1-A^2)} \tag{4.2}$$

and

$$c^2 = \tau k \frac{(1-A^2)}{(1+A^2)}. \tag{4.3}$$

We note that c is a decreasing function of wave steepness and that the highest wave contains a trapped bubble in the trough. The maximum value in this case is given by $ak = 2.292624$.

Throughout the rest of this paper we take $\tau = 1$ and $\lambda = 2\pi$, hence $k = 1$.

5. Superharmonic perturbations

We now solve (2.11) and (2.12) for ξ, η, F and σ using (4.1) as the basic steady flow. Equation (2.10) is satisfied identically provided that we take $\beta = \frac{1}{2}c^2$. The problem reduces to an eigenvalue system which we can solve, following the method of Longuet-Higgins (1978*a*). In a recent paper Chen & Saffman (1985) considered the more general problem of three-dimensional perturbations to Crapper's (1957) solution using the method of McLean *et al.* (1981). This method requires that (4.1) be inverted to obtain Y as a function of X, expressed as a Fourier series. Unfortunately this transformation appears to be singular and does not allow consideration of any form of perturbation above $ak = 0.37$, approximately. This of course is well short of the maximum wave height. Our method overcomes this grave disadvantage but is itself restricted to two-dimensional normal-mode perturbations. Thus in this paper we take ξ, η and F to be periodic in ϕ with the same basic wavelength 2π. This will allow us to calculate the superharmonic perturbations to (4.1). We will consider subharmonic perturbations in the manner of Longuet-Higgins (1978*b*) in a subsequent paper. So we take

$$\left.\begin{aligned} \xi &= \mathrm{e}^{-\mathrm{i}\sigma t}\left\{a_0+\sum_{n=1}^{\infty}\mathrm{e}^{-n\psi/c}[a_n\cos n\theta-b_n\sin n\theta]\right\},\\ \eta &= \mathrm{e}^{-\mathrm{i}\sigma t}\left\{b_0+\sum_{n=1}^{\infty}\mathrm{e}^{-n\psi/c}[b_n\cos n\theta+a_n\sin n\theta]\right\},\\ F &= \mathrm{e}^{-\mathrm{i}\sigma t}\left\{\sum_{n=1}^{\infty}[c_n\cos n\theta+d_n\sin n\theta]\right\}, \end{aligned}\right\} \tag{5.1}$$

where there is no constant term in F since $\psi = 0$ is the surface of the unperturbed wave.

We must now embark on the lengthy calculation of substitution of (5.1) into (2.11) and (2.12). This is a straightforward exercise whose details are similar to those contained in Longuet-Higgins (1978*a*). The final comparison of coefficients of $\sin n\theta$ and $\cos n\theta$ produces an infinite system of equations in the unknown real coefficients a_n, b_n, c_n, d_n which can be written in the form

$$-\mathrm{i}\sigma \boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{x} = 0. \tag{5.2}$$

The square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ have elements which are functions only of ak and $\boldsymbol{x}$ is the column vector $(\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\})$. We solve this system by truncation at $n = N$, which means that $\boldsymbol{A}_N$ and $\boldsymbol{B}_N$ are both of order $(4N+2)\times(4N+2)$ and $\boldsymbol{x}_n$ has $(4N+2)$ elements. The layout of $\boldsymbol{A}_N$ and $\boldsymbol{B}_N$ is identical with those matrices found in Longuet-Higgins (1978*a*), that is the introduction of capillarity has no effect on their form. Thus for $N = 3$, our matrices $\boldsymbol{A}_3$ and $\boldsymbol{B}_3$ have elements arranged in similar box fashion to those in figures 5 and 6 of Longuet-Higgins (1978*a*), although the values of these elements are different. This observation is important because it means that advantage can be taken of a rearrangement of these matrices, originally due to Tanaka and contained in Longuet-Higgins (1986). This leads to a significant reduction in computation time for a given N and perhaps more significantly to an increase in the value of N and hence the number of coefficients of ξ, η and F that can be calculated in a given amount of computation time.

In fact we can rewrite (5.2) as

$$(-\mathrm{i}\sigma)\begin{pmatrix}\boldsymbol{A}_{11} & 0\\ 0 & \boldsymbol{A}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix}+\begin{pmatrix}0 & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{pmatrix} = 0, \tag{5.3}$$

where $\boldsymbol{x}_1 = (\{a_n\}, \{d_n\})^{\mathrm{T}}$ and $\boldsymbol{x}_2 = (\{b_n\}, \{c_n\})^{\mathrm{T}}$. It is then straightforward to show that

$$(\sigma^2\boldsymbol{A}_{11}+\boldsymbol{B}_{12}\boldsymbol{A}_{22}^{-1}\boldsymbol{B}_{21})\boldsymbol{x}_1 = 0, \tag{5.4}$$

$$(\sigma^2\boldsymbol{A}_{22}+\boldsymbol{B}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{B}_{12})\boldsymbol{x}_2 = 0. \tag{5.5}$$

Thus if we are only interested in the values of the complex frequency σ we can solve either (5.4) or (5.5), a considerably easier task than solving (5.2). There is another benefit to be obtained from this procedure. The submatrices $\boldsymbol{A}_{11}$, $\boldsymbol{A}_{22}$, $\boldsymbol{B}_{12}$ and $\boldsymbol{B}_{21}$ each have to be calculated explicitly in order to solve for σ in either of the above equations. It is therefore an important check on the calculation that the values of σ obtained from (5.4) should be identical with those obtained from (5.5).

In what follows, we have found convergent solutions at $N = 30$ for most values of the steepness ak. Near the maximum value of ak, we have used $N = 40$. All the computations were performed on the University of Oxford's ICL 2988 computer, using subroutines from the NAG library for operations involving the matrices.

6. Results and comparison with theory

We have plotted in figure 1 the values of $\mathrm{Re}(\sigma)$ against the steepness ak of the basic nonlinear wave. The abscissa ranges from zero up to the maximum waveheight whereas the ordinate has the arbitrary range of zero to 12. The perturbations have been labelled $n = \pm 1, \pm 2, \ldots$ following Longuet-Higgins (1978*a*). Perturbations that travel in the opposite sense to the unperturbed wave have negative values of n. As $ak \rightarrow 0$ we have

$$\sigma = |n| + |n|^{\frac{3}{2}}. \tag{6.1}$$

Positive values of n correspond to travelling in the same sense as the unperturbed wave, but because of the dispersion relation for pure capillary waves, we must take the different branch of the solution to obtain $\mathrm{Re}(\sigma) > 0$. Thus we have for positive n as $ak \rightarrow 0$

$$\sigma = -(|n| - |n|^{\frac{3}{2}}) \tag{6.2}$$

Both (6.1) and (6.2) are verified in figure 1. As ak increases, the perturbation interacts with the basic wave in such a way that $\mathrm{Re}(\sigma)$ is reduced for all values of n. This can be verified using recent results from Hogan *et al.* (1988), as follows.

Let us consider two wavetrains of capillary waves in parallel propagation. One wavetrain is slightly nonlinear and has wavenumber k_1, amplitude a_1, and speed c_1. The other wavetrain is linear and has wavenumber k_2, amplitude a_2, and speed c_2. We take $k_1 < k_2$. Thus from Hogan *et al.* (1988) we have

$$c_2 = (k_2)^{\frac{1}{2}} + \tfrac{1}{8}(a_1 k_1)^2 \frac{\sigma_2}{k_2} H(\eta), \tag{6.3}$$

where $\eta^2 = k_1/k_2$. The function $H(\eta)$ is given in Hogan *et al.* (1988, equation (6.6) and figure 2). In addition the speed of wave 1 is also affected by its own amplitude according to the expression

$$c_1 = 1 - \tfrac{1}{16}(a_1 k_1)^2. \tag{6.4}$$

Now the speed of linear wave 2 in the frame of the nonlinear wave 1 is given by

$$c_2' = c_2 - c_1 \tag{6.5}$$

and so the relative frequency $\sigma_2' = k_2 c_2'$ is found from (6.3), (6.4) and (6.5) to be given by

$$\sigma_2' = -k_2 + k_2^{\frac{3}{2}} + \frac{1}{16}\left[2\frac{\sigma_2}{k_2}H(\eta) + 1\right](a_1 k_1)^2 k_2. \tag{6.6}$$

For our purposes, $k_1 = k = 1$ and $k_2 = |n|$ and we find that the frequency of wave 2 has been changed by an amount $\Delta\sigma_2$ given by

$$\Delta\sigma_2 = \frac{1}{16}\left[2|n|^{\frac{1}{2}} H\left(\frac{1}{|n|^{\frac{1}{2}}}\right) + 1\right]|n|\,(ak)^2. \tag{6.7}$$

A similar reasoning for two wavetrains in antiparallel propagation gives the result that

$$\Delta\sigma_2 = \frac{1}{16}\left[1 - 2|n|^{\frac{1}{2}} H\left(-\frac{1}{|n|^{\frac{1}{2}}}\right)\right]|n|\,(ak)^2. \tag{6.8}$$

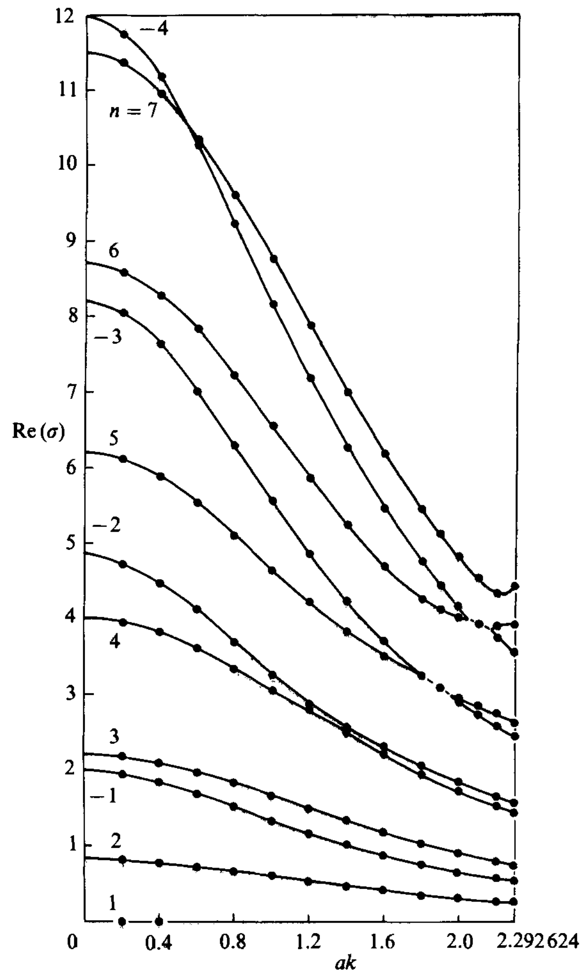

FIGURE 1. The real part of the frequency of normal mode perturbations σ plotted against the wave steepness ak of the basic wave. A continuous line denotes stability; a broken line denotes instability.

These two formulae are plotted in figure 2. The solid lines represent the actual computations and the dotted lines are either (6.7) or (6.8) depending on the sign of n. The case $n = 1$ is omitted since then the two wavelengths are equal. The case $n = 7$ would lie about directly on top of the case $n = -3$ and so has been omitted for the sake of clarity. The agreement is particularly encouraging for all values of n, even up to values of $ak = 0.4$. Indeed for the lower positive values of n, the difference is very small even at $ak = 0.6$. This gives us great confidence in the numerical code, coming on top of the other checks that have been performed.

There are other features of figure 1 that require comment, in particular the collision or near collision of various pairs of modes. Let us first consider the modes $n = 4$ and $n = -2$ near $ak = 1.4$. According to the work of MacKay & Saffman (1986), the two eigenvalues have opposite signature and so any collision may result in an instability. We have plotted in figure 3 a close-up of figure 1 near the point of interest. It appears that the two modes do not interact. We are unable to invoke

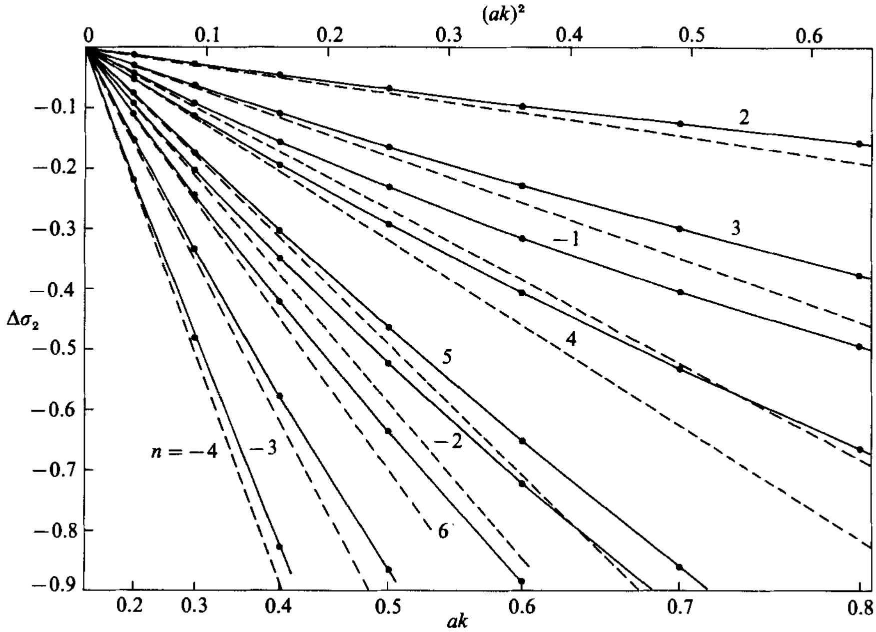

FIGURE 2. The change $\Delta\sigma_2$ of the frequency of the normal mode perturbations as a function of $(ak)^2$. A continuous line denotes the computational results; a broken line denotes the analytic expressions equation (6.7) for $n > 0$ and (6.8) for $n < 0$.

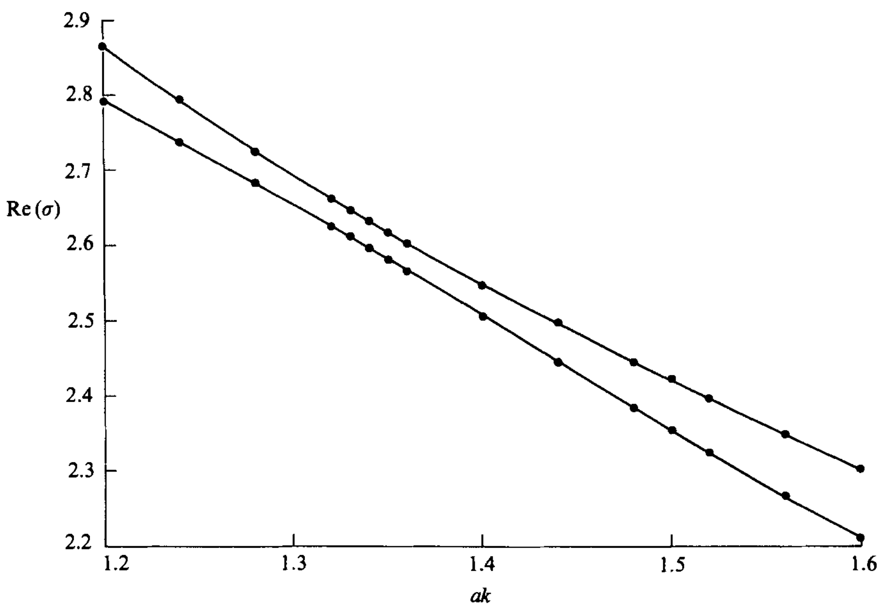

FIGURE 3. The close approach of modes $n = 4$ and $n = -2$ near $ak = 1.4$.

rounding error to make up the difference because this is of order 10^{-6}, obtained in evaluating the zero eigenvalue for the mode $n = 1$.

We now move to the clear collision of modes $n = 5$ and $n = -3$ around $ak = 1.86$. This bubble of instability has been magnified in figure 4 where both Re (σ) and Im (σ) (upper and lower curves respectively) have been plotted against ak. We can describe

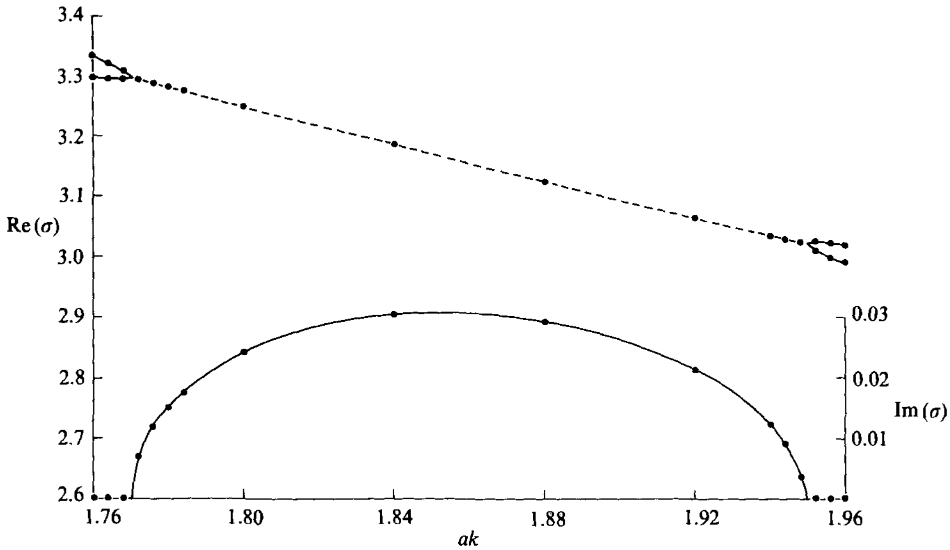

FIGURE 4. The collision of modes $n = 5$ and $n = -3$ near $ak = 1.86$. The top graph contains $\mathrm{Re}(\sigma)$ with the left-hand abscissa as a scale. The bottom graph contains the growth rate $\mathrm{Im}(\sigma)$ with the right-hand abscissa as a scale.

this instability in the notation of McLean *et al.* (1981). These authors considered perturbations to the free surface and velocity potential given by an infinite sum of modes similar to our equation (5.1). The jth perturbation has period $2\pi/q$ in the spanwise direction and period $2\pi/(p+j)$ in the propagation direction. The real numbers p and q are arbitrary. Only if p is rational will the perturbation in the propagation direction be periodic. McLean *et al.* (1981) found two classes of instability for gravity waves. Chen & Saffman (1985) outlined both classes for capillary waves. It is straightforward to show from the mode numbers that figure 4 is a Class I, $m = 4$ instability band crossing $p = 1$ at $q = 0$ with growth rate given approximately by $(ak/\pi)^8 = 0.015$. This is close to the exact computations.

In figure 5 we consider the close approach of modes $n = 7$ and $n = -4$ near $ak = 0.56$. Graphically it is impossible to tell them apart and the computed difference in the values of $\mathrm{Re}(\sigma)$ is only 5×10^{-5} at $ak = 0.560159$. We strongly suspect that instability does occur here but that our program cannot resolve the collision of the two modes. If this were the case it would be a Class II, $m = 5$ instability band crossing $p = 2$ at $q = 0$. The growth rate (and hence almost certainly the band width) is extremely small, expected to be of the order of $(ak/\pi)^{11} = 6\times10^{-9}$. Values of this size cannot be resolved by our program. The signature of the modes could be exchanged in this collision with the negative signature being transferred from the $n = -4$ eigenvalue to the mode that subsequently collides with the positive signature $n = 6$ mode near $ak = 2.1$. This is then consistent with the work of MacKay & Saffman (1986). This second collision is shown in detail in figure 6. We consider that this may be a Class I, $m = 5$ instability band passing $p = 1$ at $q = 0$, with growth rate $(ak/\pi)^{10} = 0.02$. This is in keeping with the computed values.

The above conclusions concerning the possible nature or existence of the eigenvalue collisions could be supported in another way, as sketched in figure 7. Here we show collisions for Class I and Class II instabilities in the (p, q)-plane. This is similar to figure 2 of Chen & Saffman (1985). The solid lines are valid for $ak = 0$. As ak increases,

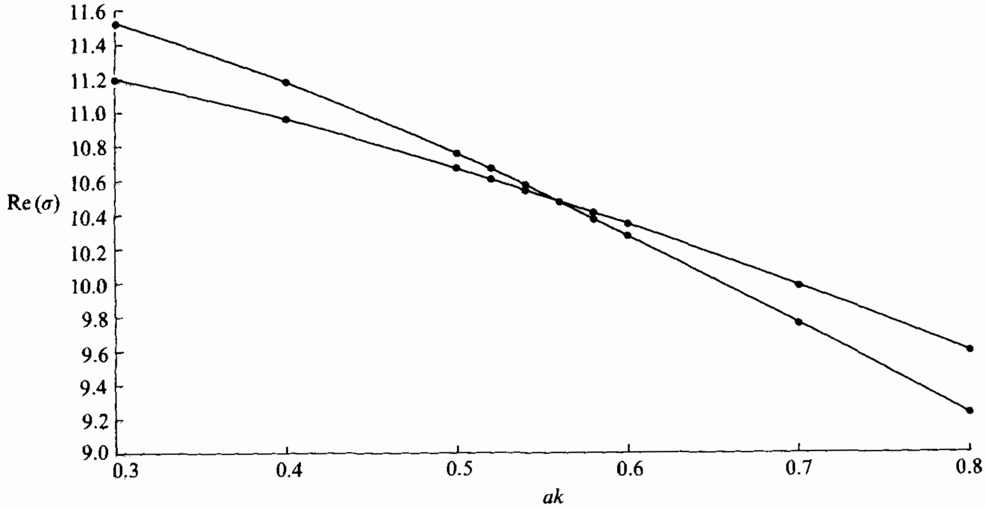

FIGURE 5. The very close approach of modes $n = 7$ and $n = -4$ near $ak = 0.56$.

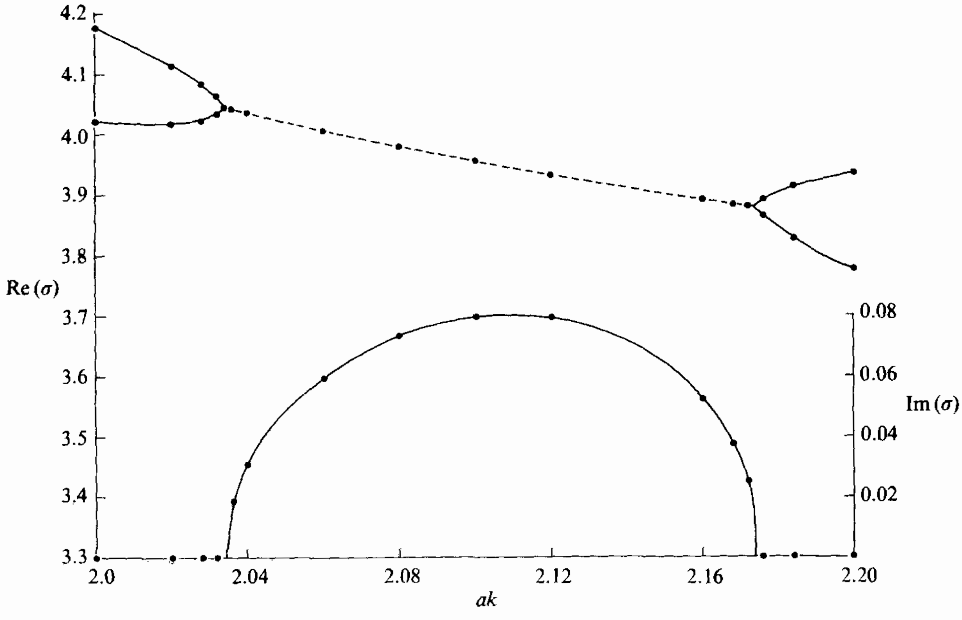

FIGURE 6. The collision of modes near $ak = 2.1$ with axes as in figure 4.

these lines will broaden (although not all at a uniform rate) and move to the left. We have sketched a possible behaviour for $ak > 0$ for the Class II, $m = 5$ instability. As it broadens and moves to the left it very soon intersects the point $(p, q) = (2, 0)$ and hence give rise to the collision between modes $n = 7$ and $n = -4$ near $ak = 0.56$. This is *not* possible for any mode in Class II with $m < 5$ because for $ak = 0$, the solid lines intersect the $q = 0$ axis at $p < 2$. Nevertheless, it may be possible for these lines to broaden and intersect the point $(p, q) = (1, 0)$ albeit at an extremely large (possibly unphysical) value of ak. Similarly it is possible that the lines of the Class I collisions would also broaden and the cases $m = 3$ and 4 intersect the point $(p, q) = (1, 0)$.

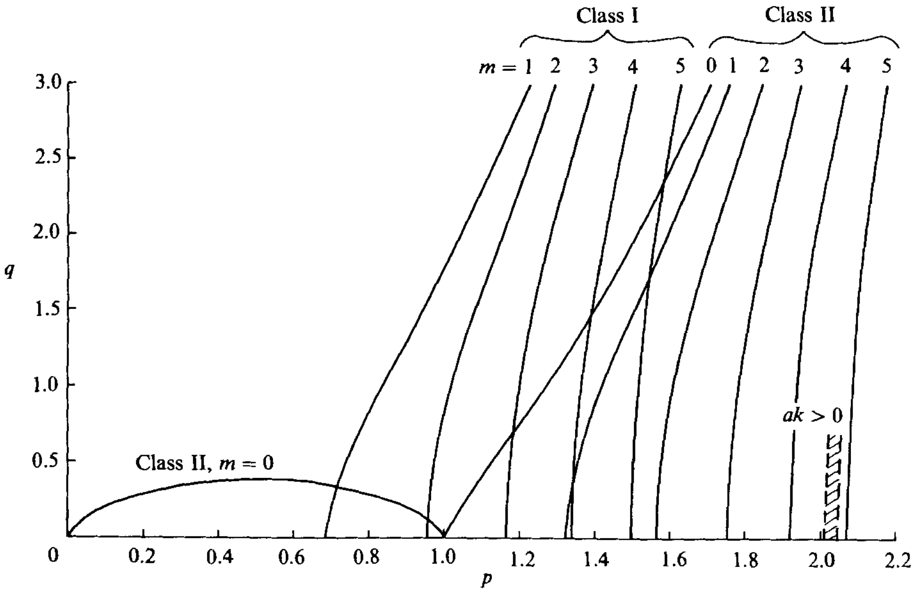

FIGURE 7. The curves for collisions at $ak = 0$ in the (p, q)-plane for Class I, $m = 1, 2, 3, 4$ and 5 instabilities and for Class II, $m = 0, 1, 2, 3, 4$ and 5 instabilities, together with a sketch of the possible behaviour of the Class II, $m = 5$ instability band for $ak > 0$.

There is however a great deal of caution to be exercised in applying the above reasoning to our computations. In fact Zhang & Melville (1987) have shown that the behaviour of these bands, albeit for gravity–capillary waves, is much more varied than has been found for pure gravity waves. For example some of the bands can broaden and then shrink again and even disappear for $ak > 0$. This gives rise to a possible explanation for the non-collision of modes $n = 4$ and $n = -2$ near $ak = 1.4$. The band of the Class I, $m = 3$ line broadens, moves to the left, detaches itself from the line $q = 0$ and subsequently disappears either before or after passing the line $p = 1$. It would then not be visible in our purely two-dimensional work. Such a behaviour is indeed exhibited in figure 4 of Zhang & Melville (1987). Between figure 4(*c*), $ak = 0.3$ and figure 4(*d*), $ak = 0.4$ the sum triad (one of the two Class II, $m = 0$ instabilities) has indeed left the $q = 0$ axis but remains visible, only to disappear completely in figure 4(*e*), $ak = 0.5$. More computation is necessary before any firm conclusions can be drawn on the behaviour of all the relevant instability bands in the (p, q)-plane.

It is quite possible to sketch the form of (4.1) for ak greater than the maximum. This corresponds to a physically unrealisable solution with parts of the surface intersecting other parts. Nevertheless we have calculated the eigenvalues for some of these values of ak, as given in figure 8. This shows two modes colliding extremely close to the maximum value of ak. (The upper mode has been omitted from figure 1.) This appears to correspond to a Class I, $m = 6$ instability near $p = 1$, $q = 0$ and fits in perfectly with the sketch given in figure 7. Nevertheless, the expected growth rate is around $(ak/\pi)^{12} = 0.03$ which is considerably smaller than our calculations. On increasing N to 50, we confirmed the values in figure 8 to three significant figures. In addition we continued our calculations beyond $ak = 2.6$ and the instability was still present. Thus it appears that it is not a bubble of instability and that the behaviour in the (p, q)-plane at these values of ak can not be so simply understood in terms of

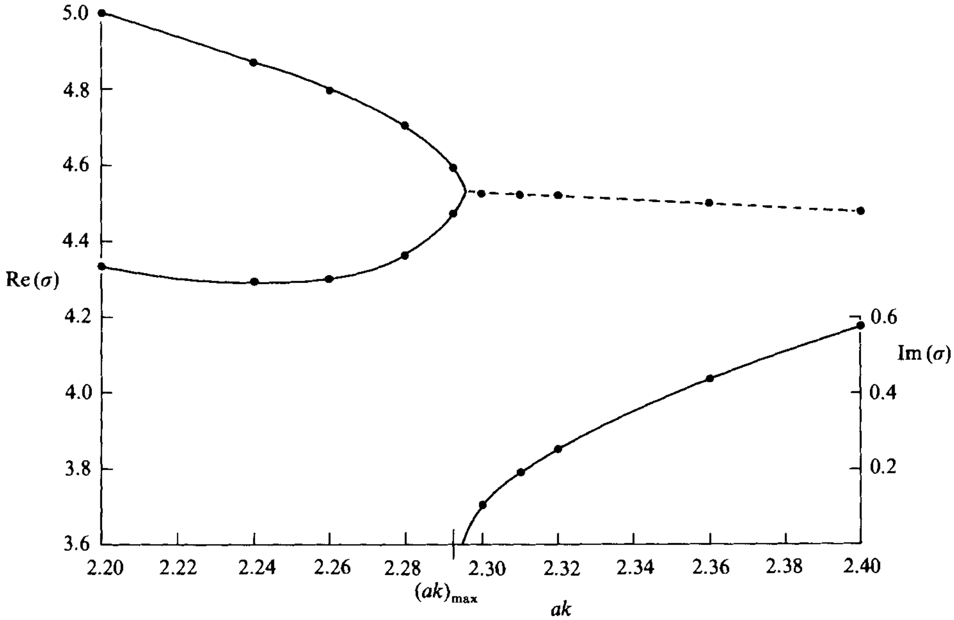

FIGURE 8. The collision of modes near the maximum physical value of ak denoted by $(ak)_{max}$ with axes as in figure 4.

linear resonances. The minimum in the lower mode at $ak = 2.24$ just before collision is correct to three significant figures.

We note that there are no extrema in the value of the rest frame wave energy E as a function of ak (Hogan 1979, figure 2). This corresponds to the fact that none of the eigenvalues with $n > 1$ have $\mathrm{Re}\,(\sigma) = 0$ and so the Tanaka (1983) superharmonic instability is absent in this case.

Finally we conclude by re-emphasizing the speculative nature of our discussions of the behaviour in the (p, q)-plane. We have presented one set of conclusions based on limited observations. Other conclusions may be possible. The resolution of the problem must await the full three-dimensional computations right up to $ak = 2.292624$ and beyond.

7. Summary

We have calculated the superharmonic normal-mode perturbations to the nonlinear capillary wave solution of Crapper (1957). By using the method of Longuet-Higgins (1978*a*), we have been able for the first time to calculate eigenvalues right up to the maximum waveheight, and beyond. Our method has been shown to produce the correct analytic solution for the perturbation of a uniform flow. In addition our computations are in excellent agreement with some recent asymptotic results for small steepness (Hogan *et al.* 1988). We have placed our results within a consistent framework for future work involving three-dimensional perturbations. We find superharmonic instabilities even in the absence of extrema in the rest frame wave energy, at relatively low waveheights. Our results for $ak > 0$ are separate from but complementary to those of Chen & Shaffman (1985) and Zhang & Melville (1987) who were both interested in subharmonic instabilities over a restricted range of waveheights. We shall extend our calculations to those pertubations in a companion paper.

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